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SOLUTION OF HEAT EQUATION ON A SEMI INFINITE LINE USING FOURIER COSINE TRANSFORM OF I-FUNCTION OF ONE VARIABLE

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ABSTRACT

In the present paper we have considered the problem of finding the temperature distribution near the end of a long rod which is insulated. For finding the solution we have derived the Fourier cosine transform for the I-function of one variable and then this transform is used to solve a boundary value problem for finding the temperature distribution near the end of the long rod which is insulated over the interval $0 < x < \infty$ and it seems that it is totally a new approach which we have adopted for solving such kind of problems

Our results are so general in nature that by suitably specializing the parameters, many more known and new results can be obtained. A few particular cases are also established in the present paper.

KEYWORDS: Fourier Cosine Transform, Saxena's I-Function of One Variable, Boundary Value Problem

1. INTRODUCTION

Our aim is to find out the solution of heat equation on a semi infinite line using Fourier Cosine Transform of I-function of one variable. Regarding the heat conduction in solids it is well known that if the temperature u in a solid body is not constant heat energy flows in the direction of the gradient $-\nabla u$ with magnitude $k|\nabla u|$ The quantity k is called the *thermal conductivity* of the material and the above principle is called *Fourier's law of heat conduction*. This law combined with the law of conservation of thermal energy, which states that *the rate of heat entering a region plus that which is generated inside the region equals the rate of heat leaving the region plus that which is stored*.

whichleads to the partial differential equation.

$$\nabla^2 u = a^{-2} u_t - q(x, y, z, t) (1.1)$$

where a^2 is another physical constant called the diffusivity. Eq. (1.1) is commonly called the *heat equation or diffusion equation*.

The French physicist J. Fourier announced in his work on heat conduction that an “arbitrary function” could be expanded in a series of sinusoidal functions. The Fourier integral was introduced by Fourier as an attempt to generalize his results from finite intervals to infinite interval. The Fourier transform while appearing in some early writings of Cauchy & Laplace, also appears in the work of Fourier.

The Fourier cosine transform of $f(x)$ is denoted by $\mathcal{F}_c\{f(x); s\}$ & defined as

$$\mathcal{F}_c\{f(x);s\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx = f_c(s), \qquad s > 0 \qquad (1.2)$$

and the inverse cosine transform is

$$\mathcal{F}_c^{-1}\{f_c(s);x\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx = f(x), \qquad x > 0 \qquad (1.3)$$

I-Function of one variable introduced by saxena V.P. [6] is defined as

$$I[x] = I_{p_i,q_i:r}^{m,n}\left[x\left|\begin{matrix}\{(a_j,\alpha_j)_{1,n}\};\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \{(b_j,\beta_j)_{1,m}\};\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right] (1.4)$$

$$= \frac{1}{2\pi\omega}\int_L \theta(\xi)x^{\xi}d\xi$$

where

$$\theta(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)}$$

$\omega = \sqrt{-1},\ p_i(i = 1,2,\ldots,r), q_i(i = 1,2,\ldots,r), m, n$ are integers satisfying $0 \le n \le p_i; 0 \le m \le q_i$; r is finite. $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive ; a_j, b_j, a_{ji}, b_{ji} are complex numbers. L is the Mellin-Barnes type contour integral which runs from $-\omega\infty$ to $+\omega\infty$ with indentations. The integral converges if

$$|\arg(x)| < \frac{\pi A_i}{2} A_i > 0, \qquad B_i < 0$$

Where

$$A_i = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ji}$$

$$B_i = \frac{1}{2}(p_i - q_i)\sum_{j=m+1}^{q_i} b_{ji} - \sum_{j=n+1}^{p_i} a_{ji}$$

2. REQUIRED RESULTS

In this section we are mentioning the results required for the evaluation of the transform and the solution of the boundary value problem.

Result I

Mellin transform of Bessel's function from Erdelyi[3]

$$\int_0^\infty x^{s-1} J_\upsilon(ax)dx = \frac{2^{s-1}\Gamma\left(\frac{s+\upsilon}{2}\right)}{a^s\Gamma\left(\frac{\upsilon}{2}-\frac{s}{2}+1\right)} (2.1)$$

Where $a > 0,\ -\mathcal{R}(\upsilon) < R(s) < \frac{3}{2}$

Result II

$$\mathcal{F}_c\{x^\upsilon e^{-ax^2}\} = \frac{1}{2} a^{-\frac{1}{2}(1+\upsilon)}\Gamma\left(\frac{1+\upsilon}{2}\right) \quad {}_1F_1\left[\frac{1+\upsilon}{2}; \frac{1}{2}; -\frac{\rho^2}{4a}\right] (2.2)$$

Where $\rho > 0$

Result III

Taking $\upsilon = -\frac{1}{2}$ in Bessel's function

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x (2.3)$$

3. FOURIER COSINE TRANSFORM OF I-FUNCTION OF ONE VARIABLE

If $\rho, \upsilon, b \in C,\ \sigma > 0$, satisfy the conditions

$$\mathcal{R}(\rho) + \sigma \min_{1\le j\le m} \mathcal{R}\left[\frac{(b_j)}{\beta_j}\right] > 0$$

And

$$\mathcal{R}(\rho) + \sigma \max_{1\le j\le n} \left[\frac{(a_j - 1)}{\alpha_j}\right] < 1$$

Then for $s, \rho > 0$ there holds the formula

$$\int_0^\infty x^{\rho-1}\cos(sx) I^{m,n}_{p_i,q_i:r}\left[bx^\sigma \left| \begin{matrix} \{(a_j,\alpha_j)_{1,n}\}; \{(a_{ji},\alpha_{ji})_{n+1,p_i}\} \\ \{(b_j,\beta_j)_{1,m}\}; \{(b_{ji},\beta_{ji})_{m+1,q_i}\} \end{matrix} \right.\right] dx$$

$$= \frac{2^{\rho-1}\sqrt{\pi}}{s^\rho} I^{m,n+1}_{p_i+2,q_i:r}\left[b\left(\frac{2}{s}\right)^\sigma \left| \begin{matrix} \left(\frac{2-\rho}{2},\frac{\sigma}{2}\right), \{(a_j,\alpha_j)_{1,n}\}; \{(a_{ji},\alpha_{ji})_{n+1,p_i}\}, \left(\frac{1-\rho}{2},\frac{\sigma}{2}\right) \\ \{(b_j,\beta_j)_{1,m}\}; \{(b_{ji},\beta_{ji})_{m+1,q_i}\} \end{matrix} \right.\right] (3.1)$$

Proof : To derive the Fourier cosine transform of I-function of one variable we start with the integral

$$I = \int_0^\infty x^{\eta-1} J_\upsilon(sx) I^{m,n}_{p_i,q_i:r}\left[bx^\sigma \left| \begin{matrix} \{(a_j,\alpha_j)_{1,n}\}; \{(a_{ji},\alpha_{ji})_{n+1,p_i}\} \\ \{(b_j,\beta_j)_{1,m}\}; \{(b_{ji},\beta_{ji})_{m+1,q_i}\} \end{matrix} \right.\right] dx$$

expressing the I-function of one variable in Mellin-Barnes integral we have

$$I=\int_0^\infty x^{\eta-1}J_\upsilon(sx)\left\{\frac{1}{2\pi\omega}\int_L \theta(\xi)b^{\xi}x^{\sigma\xi}d\xi\right\}dx$$

Changing the order of integrationunder the conditions specified and rearranging the terms, we get

$$I=\frac{1}{2\pi\omega}\int_L \theta(\xi)b^{\xi}\left\{\int_0^\infty x^{\eta+\sigma\xi-1}J_\upsilon(sx)dx\right\}d\xi$$

using the result (2.1) we obtain

$$I=2^{\eta-1}s^{-\eta}I^{m,n+1}_{p_i+2,q_i:r}\left[b\left(\frac{2}{s}\right)^{\sigma}\left|\begin{matrix}\left(1-\frac{\eta+\upsilon}{2},\frac{\sigma}{2}\right),\left\{(a_j,\alpha_j)_{1,n}\right\};\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\},\left(1-\frac{\eta-\upsilon}{2},\frac{\sigma}{2}\right)\\ \left\{(b_j,\beta_j)_{1,m}\right\};\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\}\end{matrix}\right.\right]$$

Substituting $\upsilon=-\frac{1}{2}$& using the result (2.3) & substituting $\eta-\frac{1}{2}=\rho$,we arrive at the result which provide the cosine transform of I-function of one variable which is as under

$$\int_0^\infty x^{\rho-1}\cos(sx)I^{m,n}_{p_i,q_i:r}\left[bx^{\sigma}\left|\begin{matrix}\left\{(a_j,\alpha_j)_{1,n}\right\};\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\}\\ \left\{(b_j,\beta_j)_{1,m}\right\};\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\}\end{matrix}\right.\right]dx$$

$$=\frac{2^{\rho-1}\sqrt{\pi}}{s^{\rho}}I^{m,n+1}_{p_i+2,q_i:r}\left[b\left(\frac{2}{s}\right)^{\sigma}\left|\begin{matrix}\left(\frac{2-\rho}{2},\frac{\sigma}{2}\right),\left\{(a_j,\alpha_j)_{1,n}\right\};\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\},\left(\frac{1-\rho}{2},\frac{\sigma}{2}\right)\\ \left\{(b_j,\beta_j)_{1,m}\right\};\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\}\end{matrix}\right.\right]$$

4. PROBLEMOF HEAT CONDUCTION IN SOLIDS

Let us consider the problem of finding the temperature distribution near the end of a long rod which is insulated. In such a case we might model the rod as if it were extended over the interval $0<x<\infty$ If the initial temperature distribution in the rod is $f(x)$, the problem we wish to solve is mathematically described by

$$u_{xx}=a^{-2}u_t \qquad 0<x<\infty, \qquad t>0 \qquad (4.1)$$

Boundary Condition: $u_x(0,t)\to 0, u(x,t)\to 0, u_x(x,t)\to 0\ as x\to\infty$

Initial Condition: $u(x,0)=f(x), \quad 0<x<\infty$

The fact that the interval is semi infinite together with the prescribed boundary condition at $x=0$, suggests that the Fourier cosine transform be used in this case.

Hence if we define

$$\mathcal{F}_c\{u(x,t);x\to s\}=U(s,t)\,(4.2)$$

it follows form properties of cosine transform that

$$\mathcal{F}_c\{u_{xx}(x,t); x \to s\} = -s^2U(s,t) - \sqrt{\frac{2}{\pi}}u_x(0,t) = -s^2U(s,t) \quad (4.3)$$

Also by setting $f(s) = \mathcal{F}_c\{f(x); s\}$ the transformed problem becomes

$$U_t + a^2s^2U = 0, \qquad t > 0 \qquad (4.4)$$

Initial condition $U(s,0) = f(s), \qquad 0 < s < \infty$

5. SOLUTION OF HEAT EQUATION ON A SEMI INFINITE LINE

We are providing here the solution of the boundary value problem of finding the temperature distribution near the end of the long rod which is insulated over the interval $0 < x < \infty$, and is stated below

$$u(x,t) = 2^{\rho-3/2}(a^2t)^{\rho-1/2}\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(\frac{1}{2}\right)_k k!}\left(\frac{-x^2}{4a^2t}\right)^k \times I^{m+1,n+1}_{p_{i+2},q_{i+1}:r}\left[b\left(2\sqrt{a^2t}\right)^\sigma \middle| \begin{matrix}\left(\frac{2-\rho}{2},\frac{\sigma}{2}\right),\left\{(a_j,\alpha_j)_{1,n}\right\};\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\},\left(\frac{1-\rho}{2},\frac{\sigma}{2}\right)\\ \left(\frac{1-\rho}{2}+k,\frac{\sigma}{2}\right),\left\{(b_j,\beta_j)_{1,m}\right\};\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\}\end{matrix}\right] \quad (5.1)$$

Proof: Considering the solution of (4.4) as

$$U(s,t) = f(s)e^{-a^2s^2t} \quad (5.2)$$

and applying the inverse cosine transform, we get the formal solution as

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(s)e^{-a^2s^2t}\cos sx\,ds \quad (5.3)$$

Consider $f(x) = x^{\rho-1}I^{m,n}_{p_i,q_i:r}\left[bx^\sigma \middle| \begin{matrix}\left\{(a_j,\alpha_j)_{1,n}\right\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\}\\ \left\{(b_j,\beta_j)_{1,m}\right\},\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\}\end{matrix}\right]$

Then the fourier cosine transform of $f(x)$ is as follows

$$\mathcal{F}_c\{f(x);s\} = \int_0^\infty x^{\rho-1}\cos sx\, I^{m,n}_{p_i,q_i:r}\left[bx^\sigma \middle| \begin{matrix}\left\{(a_j,\alpha_j)_{1,n}\right\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\}\\ \left\{(b_j,\beta_j)_{1,m}\right\},\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\}\end{matrix}\right]dx = f(s)$$

From (3.1) we obtain

$$f(s) = \frac{2^{\rho-1}\sqrt{\pi}}{s^\rho}I^{m,n+1}_{p_i+2,q_i:r}\left[b\left(\frac{2}{s}\right)^\sigma \middle| \begin{matrix}\left(\frac{2-\rho}{2},\frac{\sigma}{2}\right);\left\{(a_j,\alpha_j)_{1,n}\right\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\};\left(\frac{1-\rho}{2},\frac{\sigma}{2}\right)\\ \left\{(b_j,\beta_j)_{1,m}\right\},\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\}\end{matrix}\right]$$

substitute f(s) in (5.2) we get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{2^{\rho-1}\sqrt{\pi}}{s^\rho} I^{m,n+1}_{p_i+2,q_i;r}\left[b\left(\frac{2}{s}\right)^\sigma \middle| \begin{matrix} \left(\frac{2-\rho}{2},\frac{\sigma}{2}\right);\left\{(a_j,\alpha_j)_{1,n}\right\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\};\left(\frac{1-\rho}{2},\frac{\sigma}{2}\right) \\ \left\{(b_j,\beta_j)_{1,m}\right\},\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\} \end{matrix}\right]$$

$$\times e^{-a^2s^2t}\cos sx ds$$

$$= \sqrt{2}\int_0^\infty e^{-a^2s^2t}\frac{2^{\rho-1}}{s^\rho}\frac{1}{2\pi\omega}\int_L \left\{\varphi'(\xi)b^\xi\left(\frac{2}{s}\right)^{\sigma\xi}d\xi\right\}\cos sx\, ds$$

Changing the order of integrations and rearranging the terms, we get

$$= \sqrt{2}\,2^{\rho-1}\frac{1}{2\pi\omega}\int_L \varphi'(\xi)b^\xi\, 2^{\sigma\xi}\left\{\int_0^\infty e^{-a^2s^2t}s^{-\sigma\xi-\rho}\cos sxds\right\}d\xi$$

Using the result (2.2), we obtain

$$= \sqrt{2}\,2^{\rho-1}\frac{1}{2\pi\omega}\int_L \varphi'(\xi)b^\xi\, 2^{\sigma\xi}\left\{\frac{1}{2}(a^2t)^{-\frac{1}{2}(1-\sigma\xi-\rho)}\Gamma\left(\frac{1-\sigma\xi-\rho}{2}\right)\quad {}_1F_1\left[\frac{1-\sigma\xi-\rho}{2};\frac{1}{2};\frac{-x^2}{4a^2t}\right]\right\}d\xi$$

On applying the expansion of ${}_1F_1$, we have

$$= 2^{\rho-3/2}\sqrt{2}\frac{1}{2\pi\omega}\frac{(a^2t)^{\frac{\rho-1}{2}}}{2}\int_L \varphi'(\xi)b^\xi\, 2^{\sigma\xi}\,\Gamma\left(\frac{1-\sigma\xi-\rho}{2}\right)(a^2t)^{\sigma\xi/2}$$

$$\times\left\{\sum_{k=0}^\infty \frac{\Gamma\left(\frac{1-\sigma\xi-\rho}{2}+k\right)}{\Gamma\left(\frac{1-\sigma\xi-\rho}{2}\right)\left(\frac{1}{2}\right)_k k!}\left(\frac{-x^2}{4a^2t}\right)^k\right\}d\xi$$

Rearranging the terms, we obtain

$$= 2^{\rho-3/2}(a^2t)^{\rho-1/2}\sum_{k=0}^\infty \frac{1}{\Gamma\left(\frac{1}{2}\right)_k k!}\left(\frac{-x^2}{4a^2t}\right)^k$$

$$\times I^{m+1,n+1}_{p_{i+2},q_{i+1};r}\left[b\left(2\sqrt{a^2t}\right)^\sigma \middle| \begin{matrix} \left(\frac{2-\rho}{2},\frac{\sigma}{2}\right),\left\{(a_j,\alpha_j)_{1,n}\right\};\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\},\left(\frac{1-\rho}{2},\frac{\sigma}{2}\right) \\ \left(\frac{1-\rho}{2}+k,\frac{\sigma}{2}\right),\left\{(b_j,\beta_j)_{1,m}\right\};\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\} \end{matrix}\right]$$

Which is the right hand side of (5.1)

6. SPECIAL CASE

Many special cases can be found by suitably specializing the parameters one of the special case of our result is mentioned below.

Substituting $r = 1$, I-function of one variable reduces to Fox's H- function of one variable assuming $a_{j1}, \alpha_{j1}, b_{j1}, \beta_{j1}$ as $a_j, \alpha_j, b_j, \beta_j$ respectively.

$$u(x,t) = 2^{\rho-3/2}(a^2t)^{\rho-1/2}\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(\frac{1}{2}\right)_k k!}\left(\frac{-x^2}{4a^2t}\right)^k \times H_{P+2,Q+1}^{m+1,n+1}\left[b\left(2\sqrt{a^2t}\right)^{\sigma}\middle|\begin{matrix}\left(\frac{2-\rho}{2},\frac{\sigma}{2}\right),\left\{\left(a_j,\alpha_j\right)_{1,P}\right\},\left(\frac{1-\rho}{2},\frac{\sigma}{2}\right)\\ \left(\frac{1-\rho}{2}+k,\frac{\sigma}{2}\right),\left\{\left(b_j,\beta_j\right)_{1,Q}\right\}\end{matrix}\right] \quad (6.1)$$

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